Group actions, revisited

Recall: If a group G acts on a nonempty set A, then for every $g \in G$, the function

$$\sigma_g : A \to A \quad \text{defined} \quad \sigma_g(a) = g \cdot a$$

is a bijection, and the function

$$\varphi : G \to S_A \quad \text{defined} \quad \varphi(g) = \sigma_g$$

is a homomorphism. The kernel of φ is called the <u>kernel</u> of the action, and the action is <u>faithful</u> if the kernel is the identity.

Note: The kernel of the action K is a normal subgroup and we can give G/K an action on A as follows:

$$gK \cdot a = g \cdot a.$$

Check this is well-defined : If $gK = hK$ then $g = hk$ for some $k \in K$.

$$\implies g \cdot a = hk \cdot a = h \cdot (\underset{\uparrow}{k \cdot a}) = h \cdot a$$

$$\text{acts as identity}$$

Then $G/K \cong \text{im}\,\varphi \leq S_A$, so it has trivial kernel, so it's faithful!

Group actions from maps to S_A

We can also get group actions from morphisms to S_A. i.e. if A is a set and G a group s.t.

$\varphi: G \to S_A$ is a homomorphism, define the group action of G on A as follows:

For $g \in G$, $a \in A$, $g \cdot a = \underbrace{\varphi(g)}_{\substack{\text{This is} \\ \text{a bijection} \\ A \to A}} (a)$

This is in fact an action (axioms are easy to check), and all actions of G on A arise in this way. i.e.

Prop: There is a bijection between the actions of G on A and the homomorphisms $G \to A$.

Def: If G is a group, a <u>permutation representation</u> of G is any homomorphism $G \to S_A$, for some nonempty set A, thus given an action of G on A.

Orbits and stabilizers

Let G be a group acting on a set A. Recall from a HW problem that we can define an equivalence relation on A as follows:

$$a \sim b \iff a = g \cdot b, \text{ some } g \in G.$$

We already showed this is an equivalence relation, and the equivalence classes are called <u>orbits</u>. For $a \in A$, the equivalence class containing a is called the orbit of a.

Note that the stabilizer of a, $G_a := \{g \in G \mid g \cdot a = a\}$ gives us no additional elements of the orbit of a, and in fact, each coset of G_a gives us an additional element of the orbit of a. That is,

Prop: For $a \in A$, the number of elements in the orbit of a is $|G : G_a|$.

Pf: We show that there is a bijection between the cosets of G_a and the elts of the orbit of a. Call the orbit $\mathcal{O}_a$.

We define a map $\mathcal{O}_a \longrightarrow$ cosets of G_a by

$$b = g \cdot a \longmapsto g\, G_a.$$

This is well-defined: If $b = g \cdot a = h \cdot a$, then $h^{-1} g \cdot a = a \implies h^{-1} g \in G_a$. Thus, $(h^{-1} g) G_a = 1\, G_a \implies g\, G_a = h\, G_a$.

This is clearly surjective, and it's injective since if $g\, G_a = h\, G_a$, then $h^{-1} g \in G_a \implies h^{-1} g \cdot a = a \implies g \cdot a = h \cdot a$. $\square$

Def: The action is <u>transitive</u> if there's only one orbit. i.e. if $\forall \; a, b \in A \;\; \exists g \in G$ s.t. $a = g \cdot b$.

Ex: S_n always acts transitively on $\{1, \ldots, n\}$: if $a, b \in \{1, \ldots, n\}$, $(a\, b) \cdot a = b$.